

# Classification of quantum groups and Belavin–Drinfeld cohomologies

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## Abstract

In the present article we discuss the classification of quantum groups whose quasi-classical limit is a given simple complex Lie algebra  $\mathfrak{g}$ . This problem reduces to the classification of all Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$ , where  $\mathbb{K} = \mathbb{C}((\hbar))$ . The associated classical double is of the form  $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ , where  $A$  is one of the following:  $\mathbb{K}[\varepsilon]$ , where  $\varepsilon^2 = 0$ ,  $\mathbb{K} \oplus \mathbb{K}$  or  $\mathbb{K}[j]$ , where  $j^2 = \hbar$ . The first case relates to quasi-Frobenius Lie algebras. In the second and third cases we introduce a theory of Belavin–Drinfeld cohomology associated to any non-skewsymmetric  $r$ -matrix from the Belavin–Drinfeld list [1]. We prove a one-to-one correspondence between gauge equivalence classes of Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  and cohomology classes (in case II) and twisted cohomology classes (in case III) associated to any non-skewsymmetric  $r$ -matrix.

## 1 Introduction

Let  $k$  be a field of characteristic 0. According to [3], a quantized universal enveloping algebra (or a quantum group) is a topologically free topological Hopf algebra  $H$  over the formal power series ring  $k[[\hbar]]$  such that  $H/\hbar H$  is isomorphic to the universal enveloping algebra of a Lie algebra  $\mathfrak{g}$  over  $k$ .

The quasi-classical limit of a quantum group is a Lie bialgebra. By definition, a Lie bialgebra is a Lie algebra  $\mathfrak{g}$  together with a cobracket  $\delta$  which is compatible with the Lie bracket. Given a quantum group  $H$ , with comultiplication  $\Delta$ , the quasi-classical limit of  $H$  is the Lie bialgebra  $\mathfrak{g}$  of primitive elements of  $H/\hbar H$  and the cobracket is the restriction of the map  $(\Delta - \Delta^{21})/\hbar(\text{mod } \hbar)$  to  $\mathfrak{g}$ .

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The operation of taking the semiclassical limit is a functor  $SC : QUE \rightarrow LBA$  between categories of quantum groups and Lie bialgebras over  $k$ . The quantization problem raised by Drinfeld aims at finding a quantization functor, i.e. a functor  $Q : LBA \rightarrow QUE$  such that  $SC \circ Q$  is isomorphic to the identity. Moreover, a quantization functor is required to be universal, in the sense of props.

The existence of universal quantization functors was proved by Etingof and Kazhdan [4, 5]. They used Drinfeld's theory of associators to construct quantization functors for any field  $k$  of characteristic zero. Drinfeld introduced the notion of associator in relation to the theory of quasi-triangular quasi-Hopf algebras and showed that associators exist over any field  $k$  of characteristic zero. Etingof and Kazhdan proved that for any fixed associator over  $k$  one can construct a universal quantization functor. More precisely, let  $(\mathfrak{g}, \delta)$  be a Lie bialgebra over  $k$ . Then one can associate a Lie bialgebra  $\mathfrak{g}_\hbar$  over  $k[[\hbar]]$  defined as  $(\mathfrak{g} \otimes_k k[[\hbar]], \hbar\delta)$ . According to Theorem 2.1 of [5] there exists an equivalence  $\widehat{Q}$  between the category  $LBA_0(k[[\hbar]])$  of topologically free over  $k[[\hbar]]$  Lie bialgebras with  $\delta = 0 \pmod{\hbar}$  and the category  $HA_0(k[[\hbar]])$  of topologically free Hopf algebras cocommutative modulo  $\hbar$ . Moreover, for any  $(\mathfrak{g}, \delta)$  over  $k$ , one has the following:  $\widehat{Q}(\mathfrak{g}_\hbar) = U_\hbar(\mathfrak{g})$ .

The aim of the present article is the classification of quantum groups whose quasi-classical limit is a given simple complex Lie algebra  $\mathfrak{g}$ . Due to the equivalence between  $HA_0(\mathbb{C}[[\hbar]])$  and  $LBA_0(\mathbb{C}[[\hbar]])$ , this problem is equivalent to classification of Lie bialgebra structures on  $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$ . For simplicity, denote  $\mathbb{O} := \mathbb{C}[[\hbar]]$ ,  $\mathbb{K} := \mathbb{C}((\hbar))$ ,  $\mathfrak{g}(\mathbb{O}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{O}$  and  $\mathfrak{g}(\mathbb{K}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$ .

On the other hand, in order to classify cobrackets on  $\mathfrak{g}(\mathbb{O})$  it is enough to classify cobrackets on  $\mathfrak{g}(\mathbb{K})$ . Indeed, if  $\delta$  is a Lie bialgebra structure on  $\mathfrak{g}(\mathbb{O})$ , then it can be naturally extended to  $\mathfrak{g}(\mathbb{K})$ . Conversely, given a Lie bialgebra structure  $\bar{\delta}$  on  $\mathfrak{g}(\mathbb{K})$ , then by multiplying  $\bar{\delta}$  by an appropriate power of  $\hbar$ , the restriction of  $\bar{\delta}$  to  $\mathfrak{g}(\mathbb{O})$  is a Lie bialgebra structure on  $\mathfrak{g}(\mathbb{O})$ .

Now, from the general theory of Lie bialgebras it is known that each Lie bialgebra structure  $\delta$  on a fixed Lie algebra  $L$  one can construct the corresponding classical double  $D(L, \delta)$  which is the vector space  $L \oplus L^*$  together with a bracket which is induced by the bracket and cobracket of  $L$ , and a nondegenerate invariant bilinear form. We consider  $L = \mathfrak{g}(\mathbb{K})$  and prove Prop. 2.1 which states that there exists an associative, unital, commutative algebra  $A$ , of dimension 2 over  $\mathbb{K}$ , such that  $D(\mathfrak{g}(\mathbb{K}), \delta) \cong \mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ . In Prop. 2.3 we show that there are three possibilities for  $A$ :  $A = \mathbb{K}[\varepsilon]$ , where  $\varepsilon^2 = 0$ ,  $A = \mathbb{K} \oplus \mathbb{K}$  or  $A = \mathbb{K}[j]$ , where  $j^2 = \hbar$ .

Due to the correspondence Lie bialgebras–Manin triples, to any Lie bial-

gebra structure  $\delta$  on  $L$  one can associate a certain Lagrangian subalgebra  $W$  of  $D(L, \delta)$  which is complementary to  $L$  and conversely, any such  $W$  produces a Lie cobracket on  $L$ . The main problem is to obtain a classification of all such subalgebras  $W$  for the three choices of  $A$  as above. We investigate separately each choice of  $A$ .

For  $A = \mathbb{K}[\varepsilon]$ , where  $\varepsilon^2 = 0$ , it turns out that the classification problem is related to that of quasi-Frobenius Lie subalgebras over  $A$ .

In the case of  $A = \mathbb{K} \oplus \mathbb{K}$ , we introduce Belavin–Drinfeld cohomologies. Namely, for any non-skewsymmetric constant  $r$ -matrix  $r_{BD}$  from the Belavin–Drinfeld list [1], we associate a cohomology set  $H_{BD}^1(r_{BD})$ . We prove that there exists a one-to-one correspondence between any Belavin–Drinfeld cohomology and gauge equivalence classes of Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$ . Then we restrict our discussion to  $\mathfrak{g} = \mathfrak{sl}(n)$  and we show that all cohomologies are trivial. We also discuss the case of orthogonal algebras  $\mathfrak{g} = \mathfrak{o}(n)$ , where it turns out that the cohomology associated to the Drinfeld–Jimbo  $r$ -matrix is trivial for even  $n$  and non-trivial for odd  $n$ .

We finally move to the classification of Lie bialgebras whose classical double is isomorphic to  $\mathfrak{g}(\mathbb{K}[j])$ , with  $j^2 = \hbar$ . We restrict ourselves to  $\mathfrak{g} = \mathfrak{sl}(n)$  and we show that in this case a cohomology theory can be introduced too. Our result states that there exists a one-to-one correspondence between Belavin–Drinfeld twisted cohomology and gauge equivalence classes of Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$ . We prove that the twisted cohomology corresponding to the Drinfeld–Jimbo  $r$ -matrix is trivial. We also illustrate an example where the twisted cohomology corresponding to another non-skewsymmetric constant  $r$ -matrix is non-trivial.

In the last section of the article we formulate a conjecture stating that the Belavin–Drinfeld cohomology associated to the Drinfeld–Jimbo  $r$ -matrix is trivial if and only if  $\mathfrak{g}$  is simply laced. We also define the quantum Belavin–Drinfeld cohomology and formulate a second conjecture about the existence of a natural correspondence between classical and quantum cohomologies.

## 2 Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$

Let  $\mathfrak{g}$  be a simple complex finite-dimensional Lie algebra. Consider the Lie algebras  $\mathfrak{g}(\mathbb{O}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{O}$  and  $\mathfrak{g}(\mathbb{K}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$ .

We have seen that the classification of quantum groups with quasi-classical limit  $\mathfrak{g}$  is equivalent to the classification of all Lie bialgebra structures on  $\mathfrak{g}(\mathbb{O})$ . Moreover, as explained in the introduction, in order to classify Lie bialgebra structures on  $\mathfrak{g}(\mathbb{O})$ , it is enough to classify them on  $\mathfrak{g}(\mathbb{K})$ .

Let us assume that  $\bar{\delta}$  is a Lie bialgebra structure on  $\mathfrak{g}(\mathbb{K})$ . This cobracket

endows the dual of  $\mathfrak{g}(\mathbb{K})$  with a Lie bracket. Then one can construct the corresponding classical double  $D(\mathfrak{g}(\mathbb{K}), \bar{\delta})$ . As a vector space,  $D(\mathfrak{g}(\mathbb{K}), \bar{\delta}) = \mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})^*$ . As a Lie algebra, it is endowed with a bracket which is induced by the bracket and cobracket of  $\mathfrak{g}(\mathbb{K})$ . Moreover the canonical symmetric nondegenerate bilinear form on this space is invariant.

Similarly to Lemma 2.1 from [6], one can prove that  $D(\mathfrak{g}(\mathbb{K}), \bar{\delta})$  is a direct sum of regular adjoint  $\mathfrak{g}$ -modules. Combining this result with Prop. 2.2 from [2], it follows that

**Proposition 2.1.** *There exists an associative, unital, commutative algebra  $A$ , of dimension 2 over  $\mathbb{K}$ , such that  $D(\mathfrak{g}(\mathbb{K}), \bar{\delta}) \cong \mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ .*

*Remark 2.2.* The symmetric invariant nondegenerate bilinear form  $Q$  on  $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$  is given in the following way. For arbitrary elements  $f_1, f_2 \in \mathfrak{g}(\mathbb{K})$  and  $a, b \in A$  we have  $Q(f_1 \otimes a, f_2 \otimes b) = K(f_1, f_2) \cdot t(ab)$ , where  $K$  denotes the Killing form on  $\mathfrak{g}(\mathbb{K})$  and  $t : A \rightarrow \mathbb{K}$  is a trace function.

Let us investigate the algebra  $A$ . Since  $A$  is unital and of dimension 2 over  $\mathbb{K}$ , one can choose a basis  $\{e, 1\}$ , where 1 denotes the unit. Moreover, there exist  $p$  and  $q$  in  $\mathbb{K}$  such that  $e^2 + pe + q = 0$ . Let  $\Delta = p^2 - 4q \in \mathbb{K}$ . We distinguish the following cases:

- (i) Assume  $\Delta = 0$ . Let  $\varepsilon := e + \frac{p}{2}$ . Then  $\varepsilon^2 = 0$  and  $A = \mathbb{K}\varepsilon \oplus \mathbb{K} = \mathbb{K}[\varepsilon]$ .
- (ii) Assume  $\Delta \neq 0$  and has even order as an element of  $\mathbb{K}$ . This implies that  $\Delta = \hbar^{2m}(a_0 + a_1\hbar + a_2\hbar^2 + \dots)$ , where  $m$  is an integer,  $a_i$  are complex coefficients and  $a_0 \neq 0$ .

One can easily check that the equation  $x^2 = a_0 + a_1\hbar + a_2\hbar^2 + \dots$  has two solutions  $\pm x = x_0 + x_1\hbar + x_2\hbar^2 + \dots$  in  $\mathbb{O}$ .

Then  $e = -\frac{p}{2} \pm \frac{\hbar^m x}{2}$ , which implies that  $e \in \mathbb{K}$  and  $A = \mathbb{K} \oplus \mathbb{K}$ .

- (iii) Assume  $\Delta \neq 0$  and has odd order as an element of  $\mathbb{K}$ . We have  $\Delta = \hbar^{2m+1}(a_0 + a_1\hbar + a_2\hbar^2 + \dots)$ , where  $m$  is an integer,  $a_i$  are complex coefficients and  $a_0 \neq 0$ .

Again the equation  $x^2 = a_0 + a_1\hbar + a_2\hbar^2 + \dots$  has two solutions  $\pm x = x_0 + x_1\hbar + x_2\hbar^2 + \dots$  in  $\mathbb{O}$ . Since  $a_0 \neq 0$ , we have  $x_0 \neq 0$  and thus  $x$  is invertible in  $\mathbb{O}$ .

Let  $j = \hbar^{-m}(2e + p)x^{-1}$ . Then  $e^2 + pe + q = 0$  is equivalent to  $j^2 = \hbar$ .

On the other hand,  $A = \mathbb{K}e \oplus \mathbb{K}$  and  $2e = \hbar^m x j - p$  imply that  $A = \mathbb{K}f \oplus \mathbb{K}$ . Therefore, we obtain that  $A = \mathbb{K}[j]$  where  $j^2 = \hbar$ .

We can summarize the above facts:

**Proposition 2.3.** *Let  $\bar{\delta}$  be an arbitrary Lie bialgebra structure on  $\mathfrak{g}(\mathbb{K})$ . The classical double  $D(\mathfrak{g}(\mathbb{K}), \bar{\delta})$  is isomorphic to  $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ , where  $A = \mathbb{K}[\varepsilon]$  and  $\varepsilon^2 = 0$ ,  $A = \mathbb{K} \oplus \mathbb{K}$  or  $A = \mathbb{K}[j]$  and  $j^2 = \hbar$ .*

On the other hand, it is well-known, see for instance [3], that there is a one-to-one correspondence between Lie bialgebra structures on a Lie algebra  $L$  and Manin triples  $(D(L), L, W)$ . For  $L = \mathfrak{g}(\mathbb{K})$ , this fact implies the following

**Proposition 2.4.** *There exists a one-to-one correspondence between Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  for which the classical double is  $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$  and Lagrangian subalgebras  $W$  of  $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ , with respect to the nondegenerate bilinear form  $Q$ , and transversal to  $\mathfrak{g}(\mathbb{K})$ .*

**Corollary 2.5.** (i) *There exists a one-to-one correspondence between Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  for which the classical double is  $\mathfrak{g}(\mathbb{K}[\varepsilon])$ ,  $\varepsilon^2 = 0$ , and Lagrangian subalgebras  $W$  of  $\mathfrak{g}(\mathbb{K}[\varepsilon])$ , and transversal to  $\mathfrak{g}(\mathbb{K})$ .*

(ii) *There exists a one-to-one correspondence between Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  for which the classical double is  $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$  and Lagrangian subalgebras  $W$  of  $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ , and transversal to  $\mathfrak{g}(\mathbb{K})$ , embedded diagonally into  $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ .*

(iii) *There exists a one-to-one correspondence between Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  for which the classical double is  $\mathfrak{g}(\mathbb{K}[j])$ , where  $j^2 = \hbar$ , and Lagrangian subalgebras  $W$  of  $\mathfrak{g}(\mathbb{K}[j])$ , and transversal to  $\mathfrak{g}(\mathbb{K})$ .*

### 3 Lie bialgebra structures in Case I

Here we study the Lie bialgebra structures  $\delta$  on  $\mathfrak{g}(\mathbb{K})$  for which the corresponding Drinfeld double is isomorphic to  $\mathfrak{g}(\mathbb{K}[\varepsilon])$ ,  $\varepsilon^2 = 0$ . Our problem is to find all subalgebras  $W$  of  $\mathfrak{g}(\mathbb{K}[\varepsilon])$  satisfying the following conditions:

- (i)  $W \oplus \mathfrak{g}(\mathbb{K}) = \mathfrak{g}(\mathbb{K}[\varepsilon])$ .
- (ii)  $W = W^\perp$  with respect to the nondegenerate symmetric bilinear form  $Q$  on  $\mathfrak{g}(\mathbb{K}[\varepsilon])$  given by

$$Q(f_1(\hbar) + \varepsilon f_2(\hbar), g_1(\hbar) + \varepsilon g_2(\hbar)) = K(f_1, g_2) + K(f_2, g_1).$$

**Proposition 3.1.** *Any subalgebra  $W$  of  $\mathfrak{g}(\mathbb{K}[\varepsilon])$  satisfying conditions (i) and (ii) from above is uniquely defined by a subalgebra  $L$  of  $\mathfrak{g}(\mathbb{K})$  together with a nondegenerate 2-cocycle  $B$  on  $\mathfrak{g}(\mathbb{K})$ .*

*Proof.* The proof is similar to that of Th. 3.2 and Cor. 3.3 from [8]. □

*Remark 3.2.* We recall that a Lie algebra is called quasi-Frobenius if there exists a nondegenerate 2-cocycle on it. It is called Frobenius if the corresponding 2-cocycle is a coboundary. Thus we see that the classification problem for the Lagrangian subalgebras we are interested in contains the

classification of Frobenius subalgebras of  $\mathfrak{g}(\mathbb{K})$ . This question is quite complicated, as it is known from studying Frobenius subalgebras of  $\mathfrak{g}$ . However, for  $\mathfrak{g} = sl(2)$  there is only one Frobenius subalgebra, the standard parabolic one.

## 4 Lie bialgebra structures in Case II

Our task is to classify Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  for which the associated classical double is isomorphic to  $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ .

**Lemma 4.1.** *Any Lie bialgebra structure  $\delta$  on  $\mathfrak{g}(\mathbb{K})$  for which the associated classical double is isomorphic to  $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$  is a coboundary  $\delta = dr$  given by an  $r$ -matrix satisfying  $r + r^{21} = f\Omega$ , where  $f \in \mathbb{K}$  and  $\text{CYB}(r) = 0$ .*

We may suppose that  $f = 1$ . Naturally we want to classify all such  $r$  up to  $\text{Ad}(G(\mathbb{K}))$ -equivalence. Since we are mainly going to discuss this classification problem in case  $\mathfrak{g} = sl(n)$  and  $\mathfrak{g} = o(n)$ , we will work with  $GL(n, \mathbb{K})$ -equivalence and  $O(n, \mathbb{K})$ -equivalence respectively.

Let  $\overline{\mathbb{K}}$  denote the algebraic closure of  $\mathbb{K}$ . Any Lie bialgebra structure  $\delta$  over  $\mathbb{K}$  can be extended to a Lie bialgebra structure  $\bar{\delta}$  over  $\overline{\mathbb{K}}$ .

According to [1], Lie bialgebra structures on a simple Lie algebra over an algebraically closed field are coboundaries given by non-skewsymmetric  $r$ -matrices. These  $r$ -matrices have been classified up to  $\text{Ad}(G(\overline{\mathbb{K}}))$ -equivalence and they are given in terms of admissible triples.

Now, let  $r$  be an  $r$ -matrix corresponding to a Lie bialgebra on  $\mathfrak{g}(\mathbb{K})$ . Up to  $\text{Ad}(G(\overline{\mathbb{K}}))$ -equivalence, we have the Belavin–Drinfeld classification. We may therefore assume that our  $r$ -matrix is of the form  $r_X = (\text{Ad}_X \otimes \text{Ad}_X)(r_{BD})$ , where  $X \in G(\overline{\mathbb{K}})$  and  $r_{BD}$  satisfies the system  $r + r^{21} = \Omega$  and  $\text{CYB}(r) = 0$ . The corresponding bialgebra structure is  $\delta(a) = [r_X, a \otimes 1 + 1 \otimes a]$  for any  $a \in \mathfrak{g}(\mathbb{K})$ .

Let us take an arbitrary  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ . Then we have  $(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r_X), a \otimes 1 + 1 \otimes a]$  and  $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$ , which imply that  $\sigma(r_X) = r_X + \alpha\Omega$ , for some  $\alpha \in \overline{\mathbb{K}}$ . Let us show that  $\alpha = 0$ . Indeed,  $\Omega = \sigma(\Omega) = \sigma(r_X) + \sigma(r_X^{21}) = r_X + r_X^{21} + 2\alpha\Omega$ . Thus  $\alpha = 0$  and  $\sigma(r_X) = r_X$ . Consequently, we get

$$(\text{Ad}_{X^{-1}\sigma(X)} \otimes \text{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}.$$

We recall the following

**Definition 4.2.** Let  $r$  be an  $r$ -matrix. The *centralizer*  $C(r)$  of  $r$  is the set of all  $X \in G(\overline{\mathbb{K}})$  satisfying  $(\text{Ad}_X \otimes \text{Ad}_X)(r) = r$ .

We have thus obtained that  $X^{-1}\sigma(X) \in C(r_{BD})$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ .

**Definition 4.3.** Let  $r_{BD}$  be a non-skewsymmetric  $r$ -matrix from the Belavin–Drinfeld list and  $C(r_{BD})$  its centralizer. We say that  $X \in G(\overline{\mathbb{K}})$  is a *Belavin–Drinfeld cocycle* associated to  $r_{BD}$  if  $X^{-1}\sigma(X) \in C(r_{BD})$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ .

We denote the set of Belavin–Drinfeld cocycles associated to  $r_{BD}$  by  $Z(r_{BD})$ . This set is non-empty, always contains the identity.

**Definition 4.4.** Two cocycles  $X_1$  and  $X_2$  in  $Z(r_{BD})$  are called *equivalent* if there exists  $Q \in G(\mathbb{K})$  and  $C \in C(r_{BD})$  such that  $X_1 = QX_2C$ .

**Definition 4.5.** Let  $H_{BD}^1(r_{BD})$  denote the set of equivalence classes of cocycles from  $Z(r_{BD})$ . We call this set the *Belavin–Drinfeld cohomology* associated to the  $r$ -matrix  $r_{BD}$ . The Belavin–Drinfeld cohomology is said to be *trivial* if all cocycles are equivalent to the identity, and *non-trivial* otherwise.

We make the following remarks:

*Remark 4.6.* Assume that  $X \in Z(r_{BD})$ . Then for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ ,  $\sigma(X) = XC$ , for some  $C \in C(r_{BD})$ . We get  $(\text{Ad}_{\sigma(X)} \otimes \text{Ad}_{\sigma(X)})(r_{BD}) = (\text{Ad}_X \otimes \text{Ad}_X)(r_{BD})$ . Consequently,  $(\text{Ad}_X \otimes \text{Ad}_X)(r_{BD})$  induces a Lie bialgebra structure on  $\mathfrak{g}(\mathbb{K})$ .

*Remark 4.7.* Assume that  $X_1$  and  $X_2$  in  $Z(r_{BD})$  are equivalent. Then  $X_1 = QX_2C$ , for some  $Q \in G(\mathbb{K})$  and  $C \in C(r_{BD})$ . This implies that  $(\text{Ad}_{X_1} \otimes \text{Ad}_{X_1})r_{BD} = (\text{Ad}_{QX_2} \otimes \text{Ad}_{QX_2})(r_{BD})$ . In other words the  $r$ -matrices  $(\text{Ad}_{X_1} \otimes \text{Ad}_{X_1})(r_{BD})$  and  $(\text{Ad}_{X_2} \otimes \text{Ad}_{X_2})(r_{BD})$  are gauge equivalent over  $\mathbb{K}$  via an element  $Q \in G(\mathbb{K})$ .

The above remarks imply the following result.

**Proposition 4.8.** Let  $r_{BD}$  be a nonskewsymmetric  $r$ -matrix over  $\overline{\mathbb{K}}$ . There exists a one-to-one correspondence between  $H_{BD}^1(r_{BD})$  and gauge equivalence classes of Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  with classical double  $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$  and  $\overline{\mathbb{K}}$ -isomorphic to  $\delta(r_{BD})$ .

Our next goal is to compute  $H_{BD}^1(r_{BD})$ . Let us first restrict ourselves to  $\mathfrak{g} = \mathfrak{sl}(n)$  and the cohomology associated to the Drinfeld–Jimbo  $r$ -matrix  $r_{DJ}$ .

**Lemma 4.9.** Let  $X \in GL(n, \overline{\mathbb{K}})$ . Assume that for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ ,  $X^{-1}\sigma(X) \in \text{diag}(n, \overline{\mathbb{K}})$ . Then there exist  $Q \in GL(n, \mathbb{K})$  and  $D \in \text{diag}(n, \overline{\mathbb{K}})$  such that  $X = QD$ .

*Proof.* Let  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$  and  $\sigma(X) = XD_\sigma$ , where  $D_\sigma = \text{diag}(d_1, \dots, d_n)$ . Here  $d_i$  depend on  $\sigma$ . Then  $\sigma(x_{ij}) = x_{ij}d_j$ , for any  $i, j$ .

On the other hand, in each column of  $X$  there exists a nonzero element. Let us denote these elements by  $x_{i_1 1}, \dots, x_{i_n n}$ . For  $j = 1$ ,  $\sigma(x_{i_1 1}) = x_{i_1 1}d_1$  and  $\sigma(x_{i_1 1}) = x_{i_1 1}d_1$ . These relations imply that  $\sigma(x_{i_1 1}/x_{i_1 1}) = x_{i_1 1}/x_{i_1 1}$  for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$  and thus  $x_{i_1 1}/x_{i_1 1} \in \mathbb{K}$ , for any  $i$ .

Similarly,  $x_{i_2 2}/x_{i_2 2} \in \mathbb{K}, \dots, x_{i_n n}/x_{i_n n} \in \mathbb{K}$ , for any  $i$ . Let  $Q = (k_{ij})$  be the matrix whose elements are  $k_{ij} = x_{ij}/x_{i_j j}$ , for any  $i$  and  $j$ .

Thus  $X = QD$ , where  $Q \in GL(n, \mathbb{K})$  and  $D = \text{diag}(x_{i_1 1}, \dots, x_{i_n n})$ . □

**Proposition 4.10.** *For  $\mathfrak{g} = \mathfrak{sl}(n)$ , the Belavin–Drinfeld cohomology  $H_{BD}^1(r_{DJ})$  associated to  $r_{DJ}$  is trivial.*

*Proof.* It is enough to show that any cocycle is equivalent to the identity. Let  $X = (x_{ij})$  be a cocycle from  $Z(r_{DJ})$ , i.e.  $X^{-1}\sigma(X) \in C(r_{DJ})$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ .

From [1], we recall that  $C(r_{DJ}) = \text{diag}(n, \overline{\mathbb{K}})$ . It follows that  $X^{-1}\sigma(X) \in \text{diag}(n, \overline{\mathbb{K}})$ . According to Lemma 4.9, there exists  $Q \in GL(n, \mathbb{K})$  and  $D \in \text{diag}(n, \overline{\mathbb{K}})$  such that  $X = QD$ . This proves that  $X$  is equivalent to the identity. □

It turns out that the above result is true not only for  $r_{DJ}$ . Given an arbitrary  $r$ -matrix  $r_{BD}$  from the Belavin–Drinfeld list, the corresponding cohomology is also trivial. First we will take a closer look to the centralizer  $C(r_{BD})$  of an  $r$ -matrix  $r_{BD}$ .

**Lemma 4.11.** *Let  $r_{BD}$  be an arbitrary  $r$ -matrix from the Belavin–Drinfeld list. Then  $C(r_{BD}) \subseteq \text{diag}(n, \overline{\mathbb{K}})$ .*

*Proof.* Suppose we have fixed a Cartan subalgebra  $\mathfrak{h}$  and a root system. Let  $\Omega_0$  denote the Cartan part of  $\Omega$ . Let  $e_\alpha, e_{-\alpha}, h_\alpha$  be Chevalley generators such that  $K(e_\alpha, e_{-\alpha}) = 1$ . Let  $r_{DJ}$  be the classical Drinfeld–Jimbo  $r$ -matrix:

$$r_{DJ} = \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2}\Omega_0.$$

Fix an admissible triple  $(\Gamma_1, \Gamma_2, \tau)$  and some  $s \in \mathfrak{h} \wedge \mathfrak{h}$  satisfying the system  $((\alpha - \tau(\alpha)) \otimes 1)(2s) = ((\alpha + \tau(\alpha)) \otimes 1)\Omega_0$ , for any  $\alpha \in \Gamma_1$ . To this data it corresponds a non-skewsymmetric  $r$ -matrix  $r_{BD}$  from the Belavin–Drinfeld classification [1], given by the formula

$$r_{BD} = (r_{DJ} - s) + \sum_{k \in \mathbb{N}, \gamma \in (\text{Span} \Gamma_1)^+} e_{\tau^k(\gamma)} \wedge e_{-\gamma}.$$



On the other hand, given any tensor  $r = \sum r' \otimes r'' \in \mathfrak{g}(\overline{\mathbb{K}}) \otimes \mathfrak{g}(\overline{\mathbb{K}})$ , let us consider the endomorphism  $\Phi(r)$  of  $\mathfrak{g}(\overline{\mathbb{K}})$  defined by  $\Phi(r)(x) = \sum K(x, r')r''$ . We make the following remarks:

(i)  $\Phi(r_{DJ})$  is semisimple. Indeed,  $\Phi(r_{DJ})(e_{-\alpha}) = e_{-\alpha}$ ,  $\Phi(r_{DJ})(e_{\alpha}) = 0$ ,  $\Phi(r_{DJ})(h_{\beta}) = \frac{1}{2}h_{\beta}$ .

(ii)  $\Phi(e_{\tau^k(\gamma)} \wedge e_{-\gamma})$  is nilpotent because  $\tau$  is admissible.

The set of the roots of  $\mathfrak{g}$  consists of a disjoint union of the following strings:  $\{-\tau^k(\gamma), -\tau^{k-1}(\gamma), -\gamma, \tau(\gamma), \dots, \tau^k(\gamma)\}$ . Then  $\Phi(e_{\tau^k(\gamma)} \wedge e_{-\gamma})$  acts only on this string (zero on other strings) and in the above basis the corresponding matrix is lower triangular:  $\Phi(e_{\tau^m(\gamma)} \wedge e_{-\gamma})(e_{-\tau^m(\gamma)}) = e_{-\gamma}$ ,  $\Phi(e_{\tau^m(\gamma)} \wedge e_{-\gamma})(e_{\gamma}) = e_{\tau^m(\gamma)}$ . On the rest of the string,  $\Phi(e_{\tau^m(\gamma)} \wedge e_{-\gamma})$  acts trivially.

(iii)  $C(r_{DJ}) = \text{diag}(n, \overline{\mathbb{K}})$  (see [1]). One can check that  $s$  is invariant under  $\text{diag}(n, \overline{\mathbb{K}})$  and therefore it follows that  $C(r_{DJ} - s) = \text{diag}(n, \overline{\mathbb{K}})$ .

(iv) Let  $r = \sum r' \otimes r''$  and  $X \in C(r)$ . Then  $\sum K(\text{Ad}_X(r'), a)\text{Ad}_X(r'') = \sum K(r', a)r''$ , or equivalently,  $\sum \text{Ad}_X(K(r', \text{Ad}_{X^{-1}}(a))r'') = \sum K(r', a)r''$ . We then obtain  $\text{Ad}_X\Phi(r)\text{Ad}_{X^{-1}} = \Phi(r)$ .

In conclusion,  $X \in C(r)$  if and only if  $\text{Ad}_X\Phi(r) = \Phi(r)\text{Ad}_X$ .

(v) On the other hand, because of the Jordan decomposition of  $\Phi(r_{BD})$ ,  $\text{Ad}_X$  commutes with  $\Phi(r_{BD})$  if and only if  $\text{Ad}_X$  commutes with its nilpotent part and its semisimple part. But commutation with the semisimple part, which is  $\Phi(r_{DJ})$ , implies that  $X \in \text{diag}(n, \overline{\mathbb{K}})$ . This ends the proof.  $\square$

*Remark 4.12.* The above result holds also for  $\mathfrak{o}(n)$  and the proof is similar.

For  $\mathfrak{sl}(n)$  we are now able to give the exact description of  $C(r_{BD})$ .

**Lemma 4.13.**  *$C(r_{BD})$  consists of all diagonal matrices  $T = \text{diag}(t_1, \dots, t_n)$ , such that  $t_i = s_i s_{i+1} \dots s_n$ , where  $s_i \in \overline{\mathbb{K}}$  satisfy the condition:  $s_i = s_j$  if  $\alpha_i \in \Gamma_1 \setminus \Gamma_2$  and  $\tau(\alpha_i) = \alpha_j$ .*

*Proof.* Let us assume that  $r_{BD}$  is associated to an admissible triple  $(\Gamma_1, \Gamma_2, \tau)$ , where  $\Gamma_1, \Gamma_2 \subseteq \{\alpha_1, \dots, \alpha_{n-1}\}$ . Let  $T \in C(r_{BD})$ . According to Lemma 4.11,  $T \in \text{diag}(n, \overline{\mathbb{K}})$ , therefore we put  $T = \text{diag}(t_1, \dots, t_n)$ . Now we note that  $T \in C(r_{BD})$  if and only if  $(\text{Ad}_T \otimes \text{Ad}_T)(e_{\tau^k(\alpha)} \wedge e_{-\alpha}) = e_{\tau^k(\alpha)} \wedge e_{-\alpha}$  for any  $\alpha \in \Gamma_1$  and any positive integer  $k$ .

For simplicity, let us take an arbitrary  $\alpha_i \in \Gamma_1 \setminus \Gamma_2$  and suppose that  $\tau(\alpha_i) = \alpha_j$ . We then get  $t_i t_{i+1}^{-1} = t_j t_{j+1}^{-1}$ . Denote  $s_j := t_j t_{j+1}^{-1}$  for each  $j \leq n-1$  and  $s_n = t_n$ . Then  $t_j = s_j s_{j+1} \dots s_n$  and moreover  $s_i = s_j$ .  $\square$

**Theorem 4.14.** *For  $\mathfrak{g} = \mathfrak{sl}(n)$ , the Belavin–Drinfeld cohomology associated to any  $r_{BD}$  is trivial. Any Lie bialgebra structure on  $\mathfrak{g}(\overline{\mathbb{K}})$  is of the form  $\delta(a) = [r, a \otimes 1 + 1 \otimes a]$ , where  $r$  is an  $r$ -matrix which is  $GL(n, \overline{\mathbb{K}})$ -equivalent to a non-skewsymmetric  $r$ -matrix from the Belavin–Drinfeld list.*

*Proof.* Let  $X$  be a cocycle associated to  $r_{BD}$  which is a fixed  $r$ -matrix from the Belavin–Drinfeld list. Thus  $X^{-1}\sigma(X)$  belongs to the centralizer of the  $r_{BD}$ . On the other hand, according to Lemma 4.11,  $C(r_{BD}) \subseteq \text{diag}(n, \overline{\mathbb{K}})$ .

We then obtain that for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ ,  $X^{-1}\sigma(X)$  is diagonal. By Lemma 4.9, we have a decomposition  $X = QD$ , where  $Q \in GL(n, \mathbb{K})$  and  $D \in \text{diag}(\overline{\mathbb{K}})$ . Since  $\sigma(Q) = Q$ , we have  $X^{-1}\sigma(X) = (QD)^{-1}\sigma(QD) = D^{-1}Q^{-1}Q\sigma(D) = D^{-1}\sigma(D)$ . Recall that  $X^{-1}\sigma(X) \in C(r_{BD})$ . It follows that  $D^{-1}\sigma(D) \in C(r_{BD})$ .

Let  $D = \text{diag}(d_1, \dots, d_n)$ . Then  $\text{diag}(d_1^{-1}\sigma(d_1), \dots, d_n^{-1}\sigma(d_n)) \in C(r_{BD})$ . Denote  $t_i = d_i^{-1}\sigma(d_i)$  and  $T = \text{diag}(t_1, \dots, t_n)$ . According to Lemma 4.13:  $T \in C(r_{BD})$  if and only if  $t_i t_{i+1}^{-1} = t_j t_{j+1}^{-1}$ . Equivalently,  $\sigma(d_i^{-1}d_{i+1}d_j d_{j+1}^{-1}) = d_i^{-1}d_{i+1}d_j d_{j+1}^{-1}$ . It follows that  $d_i^{-1}d_{i+1}d_j d_{j+1}^{-1} \in \mathbb{K}$ . Let  $s_i := d_i d_{i+1}^{-1}$  for any  $i$  and  $s_n = d_n$ . Then we get  $s_j s_i^{-1} \in \mathbb{K}$ .

Let us fix a root  $\alpha_{i_0} \in \Gamma_1 \setminus \Gamma_2$  and let  $\tau^j(\alpha_{i_0}) = \alpha_j$ . Then  $s_j s_{i_0}^{-1} \in \mathbb{K}$ , for any  $j$ . Denote  $k_j := s_j s_{i_0}^{-1}$ .

On the other hand,  $d_j = s_j s_{j+1} \dots s_{n-1} s_n = k_j k_{j+1} \dots k_n s_{i_0}^{n-j+1}$ . Let  $K := \text{diag}(k_1 k_2 \dots k_n, k_2 \dots k_n, \dots, k_n)$  and  $C := \text{diag}(s_{i_0}^n, s_{i_0}^{n-1}, \dots, s_{i_0})$ . Note that  $D = KC$  and  $K \in GL(n, \mathbb{K})$ . Moreover, according to Lemma 4.13,  $C \in C(r_{BD})$ .

Summing up, we have obtained that if  $X$  is any cocycle associated to  $r_{BD}$ , then  $X = QD = QKC$ , with  $QK \in GL(n, \mathbb{K})$ ,  $C \in C(r_{BD})$ . This ends the proof.  $\square$

The next step in our investigation of Belavin–Drinfeld cohomologies is for orthogonal algebras  $o(m)$ . We limit ourselves to the case of Drinfeld–Jimbo  $r$ -matrix.

**Theorem 4.15.** *Let  $\mathfrak{g} = o(m)$  and  $r_{DJ}$  be the Drinfeld–Jimbo  $r$ -matrix. Then the following hold:*

- (i) *If  $m = 2n$ , then  $H_{BD}^1(r_{DJ})$  is trivial.*
- (ii) *If  $m = 2n + 1$ , then  $H_{BD}^1(r_{DJ})$  is non-trivial and consists of two elements.*

*Proof.* (i)  $m = 2n$ . On  $\overline{\mathbb{K}}^m$  fix the bilinear form  $B(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m x_i y_{m+1-i}$ . Then  $O(m, \overline{\mathbb{K}})$  consists of all  $X$  satisfying  $X^T S X = S$ , where  $S$  is the matrix with 1 on the second diagonal and zero elsewhere.

Let  $X \in O(m, \overline{\mathbb{K}})$  be a cocycle associated to  $r_{DJ}$ . Thus  $X^{-1}\sigma(X) \in C(r_{DJ})$ . Recall that  $C(r_{DJ}) = \text{diag}(m, \overline{\mathbb{K}}) \cap O(m, \overline{\mathbb{K}})$ . Therefore  $X^{-1}\sigma(X) \in \text{diag}(m, \overline{\mathbb{K}})$ . By Lemma 4.9, one has the decomposition  $X = QD$ , where  $Q \in GL(m, \mathbb{K})$  and  $D \in \text{diag}(m, \overline{\mathbb{K}})$ . Let us write  $D = \text{diag}(d_1, \dots, d_m)$  and denote by  $\mathbf{q}_i$  the columns of  $Q$  and by  $\mathbf{x}_i$  the columns of  $X$ . Then  $X = QD$  is equivalent to  $\mathbf{x}_i = d_i \mathbf{q}_i$ .

On the other hand,  $B(\mathbf{x}_i, \mathbf{x}_j) = \delta_i^{m+1-j}$ , which implies that  $B(\mathbf{q}_i, \mathbf{q}_j)d_id_j = \delta_i^{m+1-j}$ . We get  $B(\mathbf{q}_i, \mathbf{q}_j) = 0$  if  $i+j \neq m+1$  and  $B(\mathbf{q}_i, \mathbf{q}_{m+1-i})d_id_{m+1-i} = 1$ . Let  $k_i := B(\mathbf{q}_i, \mathbf{q}_{m+1-i})$ . Since  $Q \in GL(m, \mathbb{K})$ ,  $k_i \in \mathbb{K}$ . Because  $k_i^{-1} = d_id_{m+1-i}$ , it follows that  $D = Q_1 D_1$ , where  $Q_1 = \text{diag}(k_1^{-1}, \dots, k_n^{-1}, 1, \dots, 1)$  and  $D_1 = \text{diag}(d_1 k_1, \dots, d_n k_n, d_{n+1}, \dots, d_{2n})$ . We note that  $X = (QQ_1)D_1$ ,  $QQ_1 \in O(2n, \mathbb{K})$  and  $D_1 \in C(r_{DJ})$ , which proves that  $X$  is equivalent to the identity.

(ii)  $m = 2n + 1$ . By Lemma 4.9, we may write again  $X = QD$ , where  $Q \in GL(n, \mathbb{K})$  and  $D \in \text{diag}(m, \overline{\mathbb{K}})$ .

Let  $k_i := B(\mathbf{q}_i, \mathbf{q}_{m+1-i}) \in \mathbb{K}$ . Repeating the computations as in (i), one gets  $k_i^{-1} = d_id_{m+1-i}$ . If  $i = n + 1$ ,  $d_{n+1}^2 = k_{n+1}^{-1} \in \mathbb{K}$ . This implies that either  $d_{n+1} \in \mathbb{K}$  or  $d_{n+1} \in j\mathbb{K}$ .

Suppose first that  $d_{n+1} \in \mathbb{K}$ . Consider  $Q_1 = \text{diag}(k_1^{-1}, \dots, k_n^{-1}, d_{n+1}, 1, \dots, 1)$   $D_1 = \text{diag}(d_1 k_1, \dots, d_n k_n, 1, d_{n+2}, \dots, d_{2n+1})$ . We have  $D = Q_1 D_1$  and  $D_1 \in O(2n + 1, \overline{\mathbb{K}})$ . Thus  $X = (QQ_1)D_1$ ,  $QQ_1 \in O(2n + 1, \mathbb{K})$ ,  $D_1 \in C(r_{DJ})$ , i.e.  $X$  is equivalent to the trivial cocycle.

Assume that  $d_{n+1} \in j\mathbb{K}$ . Then  $\sigma(d_{n+1}) = -d_{n+1}$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ .

Let us fix  $\mathbf{s} \in \mathbb{K}^{2n+1}$  such that  $B(\mathbf{s}, \mathbf{s}) = \hbar^{-1}$  (it always exists). In the orthogonal complement of  $\mathbb{K}\mathbf{s}$  in  $\mathbb{K}^{2n+1}$  choose a basis  $\mathbf{s}_i$  satisfying  $B(\mathbf{s}_i, \mathbf{s}_j) = \delta_i^{2n+2-j}$ . Denote by  $X_0$  the matrix with columns  $\mathbf{s}_i$  and the  $n + 1$ -th column  $j\mathbf{s}$ . Then  $X_0 \in O(2n + 1, \overline{\mathbb{K}})$ . Moreover, it is a cocycle since for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ ,  $X_0^{-1}\sigma(X_0) = \text{diag}(1, 1, \dots, -1, 1, \dots, 1)$ . Let us prove that  $X$  is equivalent to  $X_0$ .

First, recall that  $X = QQ_1 D_1$ , where  $Q_1 = \text{diag}(k_1^{-1}, \dots, k_n^{-1}, d_{n+1}, 1, \dots, 1)$  and  $D_1 = \text{diag}(d_1 k_1, \dots, d_n k_n, 1, d_{n+2}, \dots, d_{2n+1})$ .

Consider  $Q_2 := QQ_1 X_0^{-1}$ . Let us show that  $Q_2 \in SO(2n + 1, \mathbb{K})$ . It suffices to prove that  $Q_1 X_0^{-1} \in GL(2n + 1, \mathbb{K})$ , since  $X_0$  and  $QQ_1$  being orthogonal over  $\overline{\mathbb{K}}$ ,  $Q_2$  is obviously orthogonal over  $\overline{\mathbb{K}}$ . It is enough to check that for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ ,  $\sigma(Q_1 X_0^{-1}) = Q_1 X_0^{-1}$ . Or equivalently,  $X_0^{-1}\sigma(X_0) = Q_1^{-1}\sigma(Q_1)$ . But  $X_0^{-1}\sigma(X_0) = \text{diag}(1, 1, \dots, -1, 1, \dots, 1)$  and  $Q_1^{-1}\sigma(Q_1) = \text{diag}(1, 1, \dots, d_{n+1}^{-1}\sigma(d_{n+1}), 1, \dots, 1)$ . Since  $\sigma(d_{n+1}) = -d_{n+1}$ , we have the desired identity.

We thus obtained  $X = Q_2 X_0 D_1$  and  $Q_2 \in O(2n + 1, \mathbb{K})$ ,  $D_1 \in C(r_{DJ})$ .

Suppose we choose another  $\mathbf{t} \in \mathbb{K}^{2n+1}$  such that  $B(\mathbf{t}, \mathbf{t}) = \hbar^{-1}$ . Then construct a basis  $\mathbf{t}_i$  of the orthogonal complement of  $\mathbb{K}\mathbf{t}$ , satisfying  $B(\mathbf{t}_i, \mathbf{t}_j) = \delta_i^{2n+2-j}$ . Denote by  $Y_0$  the matrix with columns  $\mathbf{t}_i$  and the  $n + 1$ th column  $j\mathbf{t}$ . Then one can check that  $X_0 Y_0^{-1} \in O(2n + 1, \mathbb{K})$ . It follows that  $X_0$  and  $Y_0$  are equivalent. Therefore the Belavin–Drinfeld cohomology  $H_{BD}^1(r_{DJ})$  for  $o(2n + 1)$  has exactly two classes, the identity and that of  $X_0$ .

□

We have just seen that in case  $\mathfrak{g} = o(2n)$ , the Belavin–Drinfeld cohomology  $H_{BD}^1(r_{DJ})$  is trivial. Regarding Belavin–Drinfeld cohomology  $H_{BD}^1(r_{BD})$  for an arbitrary  $r_{BD}$ , we can give an example where this set is non-trivial. Let us denote the simple roots by  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ , for  $i < n$ ,  $\alpha_n = \epsilon_{n-1} + \epsilon_n$ , where  $\{\epsilon_i\}$  is an orthonormal basis of  $\mathfrak{h}^*$ . Let  $\Gamma_1 = \{\alpha_{n-1}\}$ ,  $\Gamma_2 = \{\alpha_n\}$  and  $\tau(\alpha_{n-1}) = \alpha_n$ . Denote by  $r_{BD}$  the  $r$ -matrix corresponding to the triple  $(\Gamma_1, \Gamma_2, \tau)$  and  $s$ , where  $s \in \mathfrak{h} \wedge \mathfrak{h}$  satisfies  $((\alpha_{n-1} - \alpha_n) \otimes 1)(2s) = ((\alpha_{n-1} + \alpha_n) \otimes 1)\Omega_0$ .

**Lemma 4.16.** *The centralizer  $C(r_{BD})$  consists of all diagonal matrices of the form  $T = \text{diag}(t_1, \dots, t_{n-1}, \pm 1, \pm 1, t_{n-1}^{-1}, \dots, t_1^{-1})$ , for arbitrary nonzero  $t_1, t_2 \in \overline{\mathbb{K}}$ .*

*Proof.* By Remark 4.12,  $C(r_{BD}) \subseteq \text{diag}(2n, \overline{\mathbb{K}}) \cap O(2n, \overline{\mathbb{K}})$ . Let  $T \in C(r_{BD})$ , where  $T = \text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})$ . Since  $T$  commutes with  $r_0$  and  $r_{DJ}$ ,  $T \in C(r_{BD})$  if and only if  $(\text{Ad}_T \otimes \text{Ad}_T)(e_{\alpha_n} \wedge e_{\alpha_{n-1}}) = e_{\alpha_n} \wedge e_{\alpha_{n-1}}$ . One can check that  $(\text{Ad}_T \otimes \text{Ad}_T)(e_{\alpha_n} \wedge e_{\alpha_{n-1}}) = t_n^{-2} e_{\alpha_n} \wedge e_{\alpha_{n-1}}$ . Therefore we get  $t_n^{-2} = 1$  and the conclusion follows.  $\square$

**Proposition 4.17.** *Let  $\mathfrak{g} = o(2n)$  and  $r_{BD}$  the  $r$ -matrix corresponding to the triple  $(\Gamma_1, \Gamma_2, \tau)$ , and some  $s \in \mathfrak{h} \wedge \mathfrak{h}$ , where  $\Gamma_1 = \{\alpha_{n-1}\}$ ,  $\Gamma_2 = \{\alpha_n\}$  and  $\tau(\alpha_{n-1}) = \alpha_n$ , and  $((\alpha_{n-1} - \alpha_n) \otimes 1)(2s) = ((\alpha_{n-1} + \alpha_n) \otimes 1)\Omega_0$ .*

*Then  $H_{BD}^1(r_{BD})$  is non-trivial.*

*Proof.* Assume that  $X^{-1}\sigma(X) \in C(r_{BD})$  for all  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ . By the above lemma,  $X^{-1}\sigma(X) = \text{diag}(t_1, \dots, t_{n-1}, \pm 1, \pm 1, t_{n-1}^{-1}, \dots, t_1^{-1})$ .

On the other hand, since  $X^{-1}\sigma(X)$  is diagonal, it follows from Theorem 4.15 that there exist  $Q \in O(2n, \mathbb{K})$  and a diagonal matrix  $D \in O(2n, \overline{\mathbb{K}})$  such that  $X = QD$ . Let  $D = \text{diag}(s_1, \dots, s_n, s_n^{-1}, \dots, s_1^{-1})$ . Since  $Q \in O(2n, \mathbb{K})$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ ,  $\sigma(Q) = Q$ . We obtain  $X^{-1}\sigma(X) = D^{-1}Q^{-1}Q\sigma(D) = D^{-1}\sigma(D)$ , which is equivalent to the following system:  $s_i^{-1}\sigma(s_i) = t_i$ , for all  $i \leq n-1$  and  $s_n^{-1}\sigma(s_n) = \pm 1$ .

Assume first that there exists  $\sigma$  such that  $\sigma(s_n) = -s_n$ . Then  $s_n \in j\mathbb{K}$ . One can check that  $X$  is equivalent to  $X_0 = \text{diag}(1, \dots, 1, j, j^{-1}, 1, \dots, 1)$  which is a non-trivial cocycle.

If  $\sigma(s_n) = s_n$  for all  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ , then  $s_n \in \mathbb{K}$ . In this case,  $D = \text{diag}(s_1, \dots, s_{n-1}, 1, 1, s_{n-1}^{-1}, \dots, s_1^{-1})\text{diag}(1, \dots, 1, s_n, s_n^{-1}, 1, \dots, 1)$ , where the first matrix is in  $C(r_{BD})$  and the second in  $O(2n, \mathbb{K})$ . This proves that  $X$  is equivalent to the identity cocycle.  $\square$

## 5 Lie bialgebra structures in Case III

Here we analyse the Lie bialgebra structures on  $\mathfrak{g}(\mathbb{K})$  for which the corresponding Drinfeld double is isomorphic to  $\mathfrak{g}(\mathbb{K}[j])$ , where  $j^2 = \hbar$ . The question is to find those subalgebras  $W$  of  $\mathfrak{g}(\mathbb{K}[j])$  satisfying the following conditions:

- (i)  $W \oplus \mathfrak{g}(\mathbb{K}) = \mathfrak{g}(\mathbb{K}[j])$ .
- (ii)  $W = W^\perp$  with respect to the nondegenerate symmetric bilinear form  $Q$  given by

$$Q(f_1(\hbar) + jf_2(\hbar), g_1(\hbar) + jg_2(\hbar)) = K(f_1, g_2) + K(f_2, g_1).$$

We will restrict our discussion to  $\mathfrak{g} = sl(n)$ . We begin with the following remark. The field  $\mathbb{K}[j]$  is endowed with a conjugation. For any element  $a = f_1 + jf_2$ , its conjugate is  $\bar{a} := f_1 - jf_2$ .

If  $A = A_1 + jB_1$  and  $B = A_2 + jB_2$  are two matrices in  $sl(n, \mathbb{K}[j])$ , then  $Q(A, B) = Tr(A_1B_2 + B_1A_2)$ , i.e. the coefficient of  $j$  in  $Tr(AB)$ .

**Lemma 5.1.** *Let  $L$  be the subalgebra of  $sl(n, \mathbb{K}[j])$  which consists of all matrices  $Z = (z_{ij})$  satisfying  $z_{ij} = \bar{z}_{n+1-i, n+1-j}$ . Then  $L$  and  $sl(n, \mathbb{K})$  are isomorphic via a conjugation of  $sl(n, \mathbb{K}[j])$ .*

*Proof.* Assume that  $Z = (z_{ij})$  satisfies  $z_{ij} = \bar{z}_{n+1-i, n+1-j}$ . Then  $Z = S\bar{Z}S$ , where  $S$  is the matrix with 1 on the second diagonal and zero elsewhere.

Choose a matrix  $X \in GL(n, \mathbb{K}[j])$  such that  $\bar{X} = XS$ . Then  $\overline{XZX^{-1}} = XS\bar{Z}SX^{-1} = XZX^{-1}$  which implies that  $XZX^{-1} \in sl(n, \mathbb{K})$ . Conversely, if  $A \in sl(n, \mathbb{K})$ , then  $Z = X^{-1}AX$  satisfies the condition  $Z = S\bar{Z}S$ .  $\square$

From now on we will identify  $sl(n, \mathbb{K})$  with  $L$ . Let us find a complementary subalgebra to  $L$  in  $sl(n, \mathbb{K}[j])$ . Let us denote by  $H$  the Cartan subalgebra of  $L$ . If we identify the Cartan subalgebra of  $sl(n, \mathbb{K}[j])$  with  $\mathbb{K}^{2(n-1)}$ , then  $H$  is a Lagrangian subspace of  $\mathbb{K}^{2(n-1)}$ . Choose a Lagrangian subspace  $H_0$  of  $\mathbb{K}^{2(n-1)}$  such that  $H_0$  has trivial intersection with  $H$ . Let  $N^+$  be the algebra of upper triangular matrices of  $sl(n, \mathbb{K}[j])$  with zero diagonal. Consider  $W_0 = H_0 \oplus N^+$ . We immediately obtain the following

**Lemma 5.2.** *The subalgebra  $W_0$  as above satisfies conditions (i) and (ii), where  $sl(n, \mathbb{K})$  is identified with  $L$  as in Lemma 5.1.*

**Proposition 5.3.** *Any Lie bialgebra structure on  $sl(n, \mathbb{K})$  for which the classical double is isomorphic to  $sl(n, \mathbb{K}[j])$  is given by an  $r$ -matrix which satisfies  $CYB(r) = 0$  and  $r + r^{21} = j\Omega$ .*

*Proof.* Let  $W_0$  be as in the above lemma. By choosing two dual bases in  $W_0$  and  $sl(n, \mathbb{K})$  respectively, one can construct the corresponding  $r$ -matrix  $r_0$  over  $\overline{\mathbb{K}}$ . It is easily seen that  $r_0$  satisfies the system  $CYB(r_0) = 0$  and  $r_0 + r_0^{21} = j\Omega$ .

Let us suppose that  $W$  is another subalgebra of  $sl(n, \mathbb{K}[j])$ , satisfying conditions (i) and (ii). Then the corresponding  $r$ -matrix over  $\overline{\mathbb{K}}$  is obtained by choosing dual bases in  $W$  and  $sl(n, \mathbb{K})$  respectively. We have  $r + r^{21} = a\Omega$  for some  $a \in \mathbb{K}[j]$ . On the other hand, the classical double of the Lie bialgebras corresponding to  $r$  and  $r_0$  is the same. This implies that  $r$  and  $r_0$  are twists of each other and therefore  $a = j$ .  $\square$

On the other hand, over  $\overline{\mathbb{K}}$ , all  $r$ -matrices are gauge equivalent to the ones from Belavin–Drinfeld list. It follows that there exists a non-skewsymmetric  $r$ -matrix  $r_{BD}$  and  $X \in GL(n, \overline{\mathbb{K}})$  such that  $r = j(\text{Ad}_X \otimes \text{Ad}_X)(r_{BD})$ .

Consider an arbitrary  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ . Since  $\delta$  is a cobracket on  $sl(n, \mathbb{K})$ ,  $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$  and  $(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r), a \otimes 1 + 1 \otimes a]$ .

At this point it is worth noticing that  $\overline{\mathbb{K}} = \mathbb{K}(\hbar^{1/p^n})$ , where  $p$  runs over the set of prime integers and  $n \in \mathbb{N}^*$ . It is known that  $\text{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \cong \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}$ , where the projective limit is taken over all subgroups of  $\mathbb{Z}$ . Clearly, the subgroup  $2\hat{\mathbb{Z}}$  acts trivially on  $\mathbb{K}[j]$ . Assume that  $\sigma \in 2\hat{\mathbb{Z}}$ . Exactly as in section 4, it follows that  $\sigma(r) = r$  and if  $r = (\text{Ad}_X \otimes \text{Ad}_X)(jr_{BD})$  with  $X \in GL(n, \overline{\mathbb{K}})$ , then  $\sigma(X) = XD(\sigma)$ .

Let us notice the following:

- (i)  $\mathbb{K}[j] = \mathbb{C}((j))$ .
- (ii)  $\text{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j]) \cong 2\hat{\mathbb{Z}} \cong \hat{\mathbb{Z}}$ , the latter isomorphism being an isomorphism of abelian groups.

It means that we can use the same arguments as in the proof of Lemma 4.9 to obtain the following result

**Lemma 5.4.** *Let  $X \in GL(n, \overline{\mathbb{K}})$ . Assume that for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$ ,  $X^{-1}\sigma(X) \in \text{diag}(n, \overline{\mathbb{K}})$ . Then there exists  $P \in GL(n, \mathbb{K}[j])$  and  $D \in \text{diag}(n, \overline{\mathbb{K}})$  such that  $X = PD$ .*

Now let us consider the action of  $\sigma_2 \in \text{Gal}(\mathbb{K}[j]/\mathbb{K})$ ,  $\sigma_2(a+bj) = a-bj := \overline{a+bj}$ . Our identities imply that  $\sigma_2(r) = r + \alpha\Omega$ , for some  $\alpha \in \overline{\mathbb{K}}$ . Let us show that  $\alpha = -j$ . Indeed, since  $r + r^{21} = j\Omega$ , we also have  $\sigma_2(r) + \sigma_2(r^{21}) = -j\Omega$ . Combining these relations with  $\sigma_2(r) = r + \alpha\Omega$ , we get  $\alpha = -j$  and therefore  $\sigma_2(r) = r - j\Omega = -r_{21}$ .

Recall now that  $r = j(\text{Ad}_X \otimes \text{Ad}_X)(r_{BD})$ . It follows that  $X \in GL(n, \overline{\mathbb{K}})$  must satisfy the identity  $(\text{Ad}_{X^{-1}\sigma_2(X)} \otimes \text{Ad}_{X^{-1}\sigma_2(X)})(r_{BD}) = r_{BD}^{21}$ . We have obtained the following

**Proposition 5.5.** Any Lie bialgebra structure on  $sl(n, \mathbb{K})$  for which the classical double is  $sl(n, \mathbb{K}[j])$  is given by an  $r$ -matrix  $r = j(\text{Ad}_X \otimes \text{Ad}_X)(r_{BD})$ , where  $r_{BD}$  is a non-skewsymmetric  $r$ -matrix from the Belavin–Drinfeld list and  $X \in GL(n, \overline{\mathbb{K}})$  satisfies  $(\text{Ad}_{X^{-1}\sigma_2(X)} \otimes \text{Ad}_{X^{-1}\sigma_2(X)})(r_{BD}) = r_{BD}^{21}$  and  $(\text{Ad}_{X^{-1}\sigma(X)} \otimes \text{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}$  for  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$ .

**Definition 5.6.** Let  $r_{BD}$  be a non-skewsymmetric  $r$ -matrix from the Belavin–Drinfeld list.  $X \in G(\overline{\mathbb{K}})$  is called a *Belavin–Drinfeld twisted cocycle* associated to  $r_{BD}$  if  $(\text{Ad}_{X^{-1}\sigma_2(X)} \otimes \text{Ad}_{X^{-1}\sigma_2(X)})(r_{BD}) = r_{BD}^{21}$  and for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$ ,  $(\text{Ad}_{X^{-1}\sigma(X)} \otimes \text{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}$ .

The set of Belavin–Drinfeld twisted cocycle associated to  $r_{BD}$  will be denoted by  $\overline{Z}(r_{BD})$ .

Now, let us restrict ourselves to the case  $r_{BD} = r_{DJ}$ . In order to continue our investigation, let us prove the following

**Lemma 5.7.** Let  $S$  be the matrix with 1 on the second diagonal and zero elsewhere. Then  $r_{DJ}^{21} = (\text{Ad}_S \otimes \text{Ad}_S)r_{DJ}$ .

*Proof.* We recall that  $r_{DJ}$  is given by the following formula:

$$r_{DJ} = \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2}\Omega_0$$

where  $\Omega_0$  is the Cartan part of  $\Omega$ .

First note that  $(\text{Ad}_S \otimes \text{Ad}_S)(e_{ij} \otimes e_{ji}) = e_{n+1-i, n+1-j} \otimes e_{n+1-j, n+1-i}$ , which is a term in  $r_{DJ}^{21}$ , if  $i > j$ . On the other hand, since  $\Omega_0$  is the Cartan part of the invariant element  $\Omega$ , we get  $(\text{Ad}_S \otimes \text{Ad}_S)\Omega_0 = \Omega_0$ . This could also be proved by using the following:  $\Omega_0 = n \sum_{i=1}^n e_{ii} \otimes e_{ii} - I \otimes I$ , where  $I$  denotes the identity matrix of  $GL(n, \mathbb{K})$ . Then the identity  $r_{DJ}^{21} = (\text{Ad}_S \otimes \text{Ad}_S)r_{DJ}$  holds. □

*Remark 5.8.*  $\overline{Z}(r_{DJ})$  is non-empty. Indeed, choose  $X \in GL(n, \mathbb{K}[j])$  such that  $\sigma_2(X) = XS$ . Then  $X \in \overline{Z}(r_{DJ})$ .

**Corollary 5.9.** Let  $X$  be a Belavin–Drinfeld twisted cocycle associated to  $r_{DJ}$ . Then  $X = PD$ , where  $P \in GL(n, \mathbb{K}[j])$  and  $D \in \text{diag}(n, \overline{\mathbb{K}})$ . Moreover,  $\sigma_2(P) = PSD_1$ , where  $D_1 \in \text{diag}(n, \mathbb{K}[j])$ .

*Proof.* Since  $X$  is a twisted cocycle, for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$ ,  $X^{-1}\sigma(X) \in C(r_{DJ})$ . Recall that  $C(r_{DJ}) = \text{diag}(n, \overline{\mathbb{K}})$ . By Lemma 5.4, we have  $X = PD$ , where  $P \in GL(n, \mathbb{K}[j])$  and  $D \in \text{diag}(n, \overline{\mathbb{K}})$ .

Lemma 5.7 implies that  $S^{-1}X^{-1}\sigma_2(X) =: D_2 \in \text{diag}(n, \overline{\mathbb{K}})$ . Equivalently,  $S^{-1}D^{-1}P^{-1}\sigma_2(P)\sigma_2(D) = D_2$ . Hence  $P^{-1}\sigma_2(P) = DSD_2\sigma_2(D^{-1})$ . Let  $D_1 := S^{-1}DSD_2\sigma_2(D^{-1}) \in \text{diag}(n, \overline{\mathbb{K}})$ . Then  $\sigma_2(P) = PSD_1$  and  $D_1 \in \text{diag}(n, \mathbb{K}[j])$ .  $\square$

**Definition 5.10.** Let  $X_1$  and  $X_2$  be two Belavin–Drinfeld twisted cocycles associated to  $r_{DJ}$ . We say that they are *equivalent* if there exists  $Q \in GL(n, \mathbb{K})$  and  $D \in \text{diag}(n, \overline{\mathbb{K}})$  such that  $X_1 = QX_2D$ .

*Remark 5.11.* Assume that  $X$  is a twisted cocycle associated to  $r_{DJ}$ . By Corollary 5.9,  $X = PD$  and is equivalent to the twisted cocycle  $P \in GL(n, \mathbb{K}[j])$ .

**Definition 5.12.** Let  $\overline{H}_{BD}^1(r_{DJ})$  denote the set of equivalence classes of twisted cocycles associated to  $r_{DJ}$ . We call this set the *Belavin–Drinfeld twisted cohomology* associated to the  $r$ -matrix  $r_{DJ}$ .

*Remark 5.13.* If  $X_1$  and  $X_2$  are equivalent, then the corresponding  $r$ -matrices  $r_1 = j(\text{Ad}_{X_1} \otimes \text{Ad}_{X_1})(r_{DJ})$  and  $r_2 = j(\text{Ad}_{X_2} \otimes \text{Ad}_{X_2})(r_{DJ})$  are gauge equivalent via  $Q \in GL(n, \mathbb{K})$ .

**Proposition 5.14.** *There is a one-to-one correspondence between  $\overline{H}_{BD}^1(r_{DJ})$  and gauge equivalence classes of Lie bialgebra structures on  $sl(n, \mathbb{K})$  with classical double  $sl(n, \mathbb{K}[j])$  and  $\overline{\mathbb{K}}$ -isomorphic to  $\delta(r_{DJ})$ .*

**Proposition 5.15.** *For  $\mathfrak{g} = sl(n)$ , the Belavin–Drinfeld twisted cohomology  $\overline{H}_{BD}^1(r_{DJ})$  is trivial.*

*Proof.* Let  $X$  be a twisted cocycle associated to  $r_{DJ}$ . By Remark 5.11,  $X$  is equivalent to a twisted cocycle  $P \in GL(n, \mathbb{K}[j])$ , associated to  $r_{DJ}$ . We may therefore assume from the beginning that  $X \in GL(n, \mathbb{K}[j])$  and it remains to prove that all such cocycles are equivalent.

Choose  $X_0 \in GL(n, \mathbb{K}[j])$  such that  $\overline{X_0} = X_0S$ . We will prove that  $X$  and  $X_0$  are equivalent, i.e.  $X = QX_0D'$ , for some  $Q \in GL(n, \mathbb{K})$  and  $D' \in \text{diag}(n, \mathbb{K}[j])$ . We will illustrate the computations for  $n = 2, 3$ . For  $n = 2$ , one can choose

$$X_0 = \begin{pmatrix} 1 & 1 \\ j & -j \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{K}[j]).$$

satisfy  $\overline{X} = XSD$  with  $D = \text{diag}(d_1, d_2) \in GL(2, \mathbb{K}[j])$ . The identity is equivalent to the following system:  $\overline{a} = bd_1$ ,  $\overline{b} = ad_2$ ,  $\overline{c} = dd_1$ ,  $\overline{d} = cd_2$ .



Assume that  $cd \neq 0$ . Let  $a/c = a' + b'j$ . Then  $b/d = a' - b'j$ . One can immediately check that  $X = QX_0D'$ , where

$$Q = \begin{pmatrix} a' & b' \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{K}), \quad D' = \text{diag}(c, d) \in \text{diag}(2, \mathbb{K}[j]).$$

Let  $n = 3$ . In this case one can choose

$$X_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ j & 0 & -j \end{pmatrix}.$$

Let  $X = (a_{ij}) \in GL(3, \mathbb{K}[j])$  satisfy  $\overline{X} = XSD$ , with  $D = \text{diag}(d_1, d_2, d_3) \in GL(3, \mathbb{K}[j])$ . The identity is equivalent to the following system:  $\overline{a_{11}} = d_1a_{13}$ ,  $\overline{a_{21}} = d_1a_{23}$ ,  $\overline{a_{31}} = d_1a_{33}$ ,  $\overline{a_{12}} = d_2a_{12}$ ,  $\overline{a_{22}} = d_2a_{22}$ ,  $\overline{a_{32}} = d_2a_{32}$ ,  $\overline{a_{13}} = d_3a_{11}$ ,  $\overline{a_{23}} = d_3a_{21}$ ,  $\overline{a_{33}} = d_3a_{31}$ . Assume that  $a_{21}a_{22}a_{23} \neq 0$ .

Let  $a_{11}/a_{21} = b_{11} + b_{13}j$  and  $a_{31}/a_{21} = b_{31} + b_{33}j$ . Then  $a_{13}/a_{23} = b_{11} - b_{13}j$  and  $a_{33}/a_{23} = b_{31} - b_{33}j$ . On the other hand, let  $b_{12} := a_{12}/a_{22}$  and  $b_{32} := a_{22}/a_{32}$ . Note that  $b_{12} \in \mathbb{K}$ ,  $b_{32} \in \mathbb{K}$ . One can immediately check that  $X = QX_0D'$ , where

$$Q = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 1 & 1 & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in GL(3, \mathbb{K}), \quad D' = \text{diag}(a_{21}, a_{22}, a_{23}) \in \text{diag}(3, \mathbb{K}[j]).$$

For  $n > 3$  the proof is quite similar to the discussed cases above.  $\square$

The next step would be to compute the Belavin–Drinfeld twisted cohomology corresponding to an arbitrary  $r$ -matrix  $r_{BD}$ . However, we do not expect this set to be trivial, as it happened for the untwisted cohomology. Actually there are cases when even  $\overline{Z}(r_{BD})$  is empty.

*Remark 5.16.* Let  $r_{BD}$  be an arbitrary  $r$ -matrix associated to an admissible triple  $(\Gamma_1, \Gamma_2, \tau)$ . Then  $r_{BD}^{21}$  is associated to the triple  $(\Gamma_2, \Gamma_1, \tau^{-1})$ . On the other hand, one can check that the  $r$ -matrix  $(\text{Ad}_S \otimes \text{Ad}_S)r_{BD}$  is associated to the triple  $(S(\Gamma_1), S(\Gamma_2), S(\tau))$ , where  $S(\Gamma_i) := \{S(\alpha) : S(\alpha)(h) = \alpha(\text{Ad}_S(h))\}$  and  $S(\tau(\alpha)) = \tau(S(\alpha))$ .

Assume  $X \in \overline{Z}(r_{BD})$ . By definition,  $(\text{Ad}_{X^{-1}\sigma(X)} \otimes \text{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}^{21}$ . It follows that  $r_{BD}^{21}$  and  $(\text{Ad}_S \otimes \text{Ad}_S)r_{BD}$  are gauge equivalent, so the corresponding admissible triples should be the same. Therefore we get  $S(\Gamma_1) = \Gamma_2$  and  $S(\tau) = \tau^{-1}$ . In conclusion, if one of these conditions is not satisfied then  $\overline{Z}(r_{BD})$  is empty.

## 6 Conjectures

### 6.1 Belavin–Drinfeld cohomology conjecture

Let  $\mathfrak{g}$  be a simple Lie algebra and  $G = \text{Ad}(\mathfrak{g})$  be the corresponding adjoint group. Let  $C(\overline{\mathbb{K}}, r_{BD})$  be the subgroup of elements of  $G(\overline{\mathbb{K}})$  which act trivially on  $r_{BD}$ .

**Definition 6.1.** We say that  $X \in G(\overline{\mathbb{K}})$  is a *Belavin–Drinfeld cocycle* associated to  $r_{BD}$  if  $X^{-1}\sigma(X) \in C(\overline{\mathbb{K}}, r_{BD})$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ .

**Definition 6.2.** Two Belavin–Drinfeld cocycles  $X_1$  and  $X_2$  are *equivalent* if  $X_1 = QX_2C$ , where  $Q \in G(\mathbb{K})$  and  $C \in C(\overline{\mathbb{K}}, r_{BD})$ .

Let us denote the set of equivalence classes by  $H_{BD}^1(G, r_{BD})$ .

**Conjecture 6.3.** Let  $\mathfrak{g}$  be a simple Lie algebra and  $r_{DJ}$  the Drinfeld–Jimbo  $r$ -matrix. Then  $H_{BD}^1(G, r_{DJ})$  is trivial if and only if  $\mathfrak{g}$  is simply laced (type A, D, E).

*Remark 6.4.* It is not difficult to deduce from Theorems 4.14, 4.15 that the conjecture is true for A and D series.

### 6.2 Quantization conjecture

Let  $L$  be a finite dimensional Lie algebra over  $\mathbb{C}$  and  $\delta$  a Lie bialgebra structure on  $L(\mathbb{K})$  such that  $\delta = 0(\text{mod } \hbar)$ .

Let  $(U_\hbar(L), \Delta_\hbar)$  be the corresponding quantum group. Let  $G(\mathbb{K}) = \text{Ad}(L(\mathbb{K}))$  and  $G(\overline{\mathbb{K}}) = \text{Ad}(L(\overline{\mathbb{K}}))$ . Let us define the centralizer  $C(\overline{\mathbb{K}}, \delta)$ .

Consider the classical double  $D(L(\mathbb{K}), \delta)$ . Clearly,  $\delta$  can be extended to a Lie bialgebra structure  $\bar{\delta}$  on  $L(\overline{\mathbb{K}})$  and  $D(L(\overline{\mathbb{K}}), \bar{\delta})$  contains  $D(L(\mathbb{K}), \delta)$ , more precisely  $D(L(\overline{\mathbb{K}}), \bar{\delta}) = D(L(\mathbb{K}), \delta) \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ . The universal classical  $r$ -matrix  $r_\delta = \sum e_i \otimes e^i$  is the same for  $D(L(\mathbb{K}), \delta)$  and  $D(L(\overline{\mathbb{K}}), \bar{\delta})$ .

**Definition 6.5.** The centralizer  $C(\overline{\mathbb{K}}, \delta)$  consists of all  $X \in G(\overline{\mathbb{K}})$  such that  $(\text{Ad}_X \otimes \text{Ad}_X^*)(r_\delta) = r_\delta + \alpha\Omega$ , where  $\Omega$  is an invariant element of  $D(L(\overline{\mathbb{K}}), \bar{\delta})^{\otimes 2}$  and  $\text{Ad}^*$  is the coadjoint representation on  $(L(\overline{\mathbb{K}}))^*$ .

**Definition 6.6.** We say that  $X \in G(\overline{\mathbb{K}})$  is a *Belavin–Drinfeld cocycle* associated to  $\delta$  if  $\sigma(X) = XC$ , where  $C \in C(\overline{\mathbb{K}}, \delta)$ .

Two cocycles, associated to  $\delta$ ,  $X_1$  and  $X_2$  are *equivalent* if  $X_1 = QX_2C$ , where  $Q \in G(\mathbb{K})$  and  $C \in C(\overline{\mathbb{K}}, \delta)$ .

The set of equivalence classes will be denoted by  $H_{BD}^1(G, \delta)$ .

Now let us define quantum Belavin–Drinfeld cohomology. The quantum group  $(U_h(L), \Delta_h)$  is defined over  $\mathbb{O} = \mathbb{C}[[\hbar]]$ . We extend the Hopf structures of  $U_h(L)$  to  $U_h(L, \mathbb{K}) = U_h(L) \otimes_{\mathbb{O}} \mathbb{K}$  and  $U_h(L, \overline{\mathbb{K}}) = U_h(L) \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ . By abuse of notation,  $\Delta_{\hbar}$  denotes all three comultiplications.

**Definition 6.7.** Let  $P$  be an invertible element of  $U_h(L, \overline{\mathbb{K}})$ . We say that it belongs to  $C(U_h(L), \Delta_{\hbar})$  if  $(P \otimes P)\Delta_{\hbar}(P^{-1}aP)(P^{-1} \otimes P^{-1}) = \Delta_{\hbar}(a)$  for all  $a \in U_h(L)$ .

Denote  $F_P := (P \otimes P)\Delta_{\hbar}(P^{-1}) \in U_h(L, \overline{\mathbb{K}})^{\otimes 2}$ .

**Definition 6.8.**  $P$  is called a *quantum Belavin–Drinfeld cocycle* if  $\sigma(P) = PC$ , for any  $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$  and some  $C \in C(U_h(L), \Delta_{\hbar})$ .

Two quantum cocycles  $P_1$  and  $P_2$  are *equivalent* if  $P_2 = QP_1C$  where  $Q$  is an invertible element of  $U_h(L, \mathbb{K})$  and  $C \in C(U_h(L), \Delta_{\hbar})$ .

*Remark 6.9.* On  $U_h(L)$  consider the comultiplications  $\Delta_{\hbar, P_1}(a) = F_{P_1}\Delta_{\hbar}(a)F_{P_1}^{-1}$  and  $\Delta_{\hbar, P_2}(a) = F_{P_2}\Delta_{\hbar}(a)F_{P_2}^{-1}$ . Clearly,  $\Delta_{\hbar, P_2}(a) = (Q \otimes Q)\Delta_{\hbar, P_1}(Q^{-1}aQ) \cdot (Q^{-1} \otimes Q^{-1})$ . Since  $Q \in U_h(L(\mathbb{K}))$ , it is natural to call  $\Delta_{\hbar, P_1}$  and  $\Delta_{\hbar, P_2}$   $\mathbb{K}$ -equivalent comultiplications on  $U_h(L(\mathbb{K}))$ .

The set of equivalence classes of quantum Belavin–Drinfeld cocycles will be denoted by  $H_{q-BD}^1(\Delta_{\hbar})$ .

**Conjecture 6.10.** *There is a natural correspondence between  $H_{BD}^1(G, \delta)$  and  $H_{q-BD}^1(\Delta_{\hbar})$ .*

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